XXI Congresso U.M.I.
Pavia, 2–7 settembre 2019

Modularity of the Minimal Model Programme

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Bibliography:

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- G. Codogni, L. Tasin and F. Viviani, "On the first steps of the minimal model program for the moduli space of stable pointed curves", arxiv, 2018
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When our goal is to understand a projective variety[1] X, a sensible option is to run the Minimal Model Programme (MMP).

A run of the MMP is a sequence of elementary birational transformations

$$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n$$

Such that each transformation can be well understood, and either the final variety X_n is a Mori Fiber Space, or its canonical line bundle K_{X_n} is nef. Each transformation is called a step of the MMP.

The sequence of birational transformations is <u>not</u> unique. Many different runs of the MMP are possible.

Let us now change topic for a few minutes

[1]to run the MMP, we need some regularity assumptions on X. The base field has to be algebraically closed of characteristic zero. The singularities of X have to be Kawamata Log Terminal (in particular,we can run the MMP when X is smooth or it has finite quotient singularities)

Another common goal in algebraic geometry is to construct a moduli space for a class of objects $\mathcal{C}$.

A moduli space M for the class $\mathcal{C}$ is a projective variety, possibly with finite quotient singularities, whose points parametrize object of $\mathcal{C}$.

Usually, it is not possible to construct the full moduli space M. However, after choosing a stability condition S, we are often able to construct the moduli space $M(S)$ of S-stable objects of $\mathcal{C}$.

We do not want to give a precise definition for the notions of "class of objects" and "stability conditions", as they will be clear in the examples that we will present. For the moment, we just think a stability condition as a meaningful condition on the objects of $\mathcal{C}$, so an object C is S-stable if something important does happen.

We now ask ourselves: what does happen if we change S? How does $M(S)$ change? And what are the all the possible choices for S?

To answer, we have to make a <u>wall-crossing</u>.

Given another stability condition T, we can look at the (usually open and dense) subset U_T of $M(S)$ consisting of T-stable objects, and to the (usually open and dense) subset V_S of $M(T)$ consisting of S-stable objects.

The quasi-projective sets U_S and V_T parametrizes objects in $\mathcal{C}$ which are both S and T

stable, so they are canonically isomorphic.

In this way, we get a birational map

$$M(S) \dashrightarrow M(T)$$

and going this way we say that we have crossed the wall which divides S from T, or that we have made a wall-crossing.

On the other hand, we can look at $M(S)$ as a projective variety, forgetting that it is a moduli space, and run the MMP.

The question that I want to address in this talk is

Do runs of the MMP match up with wall crossings?

(and would this be useful?)

Of course I can not give any answer in this general and vague set-up, but I can discuss two important examples.

The blow-up of $\mathbb{P}^4$ at 8 points and its Fano model, via vector bundles on a degree 1 del Pezzo surface

joint work with C. Casagrande and A. Fanelli

Let X be a smooth Fano manifold of dimension 4 birational to the blow-up of $\mathbb{P}^4$ at 8 points in general positions.For many reasons we are keen to study X, and we are able to do it by matching up MMP and wall-crossing.

Theorem (Mukai)**.** *There exists a del Pezzo surface S of degree 1 such that X is isomorphic to the moduli space $M(-K_S)$ of rank two vector bundles on S, with $c_1 = -K_S$, $c_2 = 2$, and stable with respect to the stability conditions*[2] $-K_S$.

In this set-up, changing stability condition means just to replace $-K_S$ with some other ample line bundle on S. In this way, we obtain other moduli spaces $M(L)$ which are naturally birational to $X = M(-K_S)$.

Our main result is the following

[2] Any ample $\mathbb{Q}$-line bundle on S defines a Gieseker-Maruyama stability condition for vector bundles on S. For the purpose of this talk, it is not important to write the definition. We just recall that, with this kind of stability condition, the moduli space is a smooth projective variety.

Theorem (Casagrande, C. , Fanelli)**.** *Given a run of MMP for X*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n$$

There are ample line bundles L_i on S such that the morphism

$$X_i \dashrightarrow X_{i+1}$$

is isomorphic to the wall-crossing morphism

$$M(L_i) \dashrightarrow M(L_{i+1})$$

for every $i = 0, \ldots, n$.

Building on this result, and on an explicit description of the maps $M(L_i) \dashrightarrow M(L_{i+1})$, we can:

- compute the Betti and Hodge numbers of X,
- describe explicitly the effective, movable and nef cones of X,
- describe the geometry of the anti-canonical and bi-anti-canonical maps,
- give an isomorphism between the automorphisms groups of S and X,
- prove that X uniquely determines S (Torelli type statement).

Before moving to the next example, let us describe explicitly one case of the map $M(L_i) \dashrightarrow M(L_{i+1})$.

Let ℓ be a -1 curve on S. Let $Wall$ be the hyperplane in the Néron-Severi group of S defined as

$$Wall := \{L \text{ such that } (K_S + 2\ell) \cdot L = 0\} .$$

Let L_1 and L_2 be two generic ample $\mathbb{Q}$-line bundles which sits one on the left hand side and the other on the right hand side of this $Wall$ [3].

The natural morphism

$$M(L_1) \dashrightarrow M(L_2)$$

is the wall-crossing obtained crossing the hyperplane $Wall$ which divides L_1 from L_2. In this case, it is a flip, so it is a step of the MMP of $X \cong M(-K_S)$.

More explicitly, this flip replace the locus

$$P_\ell := \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_S(\ell), \mathcal{O}_S(-K_S - \ell)) \cong \mathbb{P}^2 \subset M(L_1)$$

with the locus

$$Z_\ell := \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_S(-K_S - \ell), \mathcal{O}_S(\ell)) \cong \mathbb{P}^1 \subset M(L_2) .$$

Let us now move to the next and last example !

[3] More precisely, L_1 and L_2 are generic ample $\mathbb{Q}$-line bundles such that $(K_S + 2\ell)L_1 < 0$, $(K_S + 2\ell)L_2 > 0$, and they are close enough to the hyperplane $Wall$ with respect to the metric induced by the intersection pairing.

On the first steps of the minimal model program for the moduli space of stable pointed curves

joint work with L. Tasin and F. Viviani

We want to study algebraic curves and their moduli space. For simplicity, we look just at curves with genus at least two and no marked points.

The classical stability notion is Deligne-Mumford (DM) stability. By definition, a curve is DM-stable if it has ample canonical bundle and at most nodal singularities.

The moduli space $\overline{M}_g$ of genus g Deligne-Mumford stable curves is a projective variety with finite quotient singularities.

Our goals are to understand the geometry of $\overline{M}_g$,
and to describe other sensible stability notions for curves

Another classical stability notion is pseudo-stability. A curve is pseudo-stable if the canonical bundle is ample, it has at most nodes and cusps as singularities, and it has not any attached elliptic tail. A classical result is the following.

Theorem (Schubert, ...). *The moduli space of genus g-pseudo-stable curves M_g^{ps} is a projective variety with finite quotient singularities.*

The canonical morphism

$$\overline{M}_g \dashrightarrow M_g^{ps}$$

is regular, and it is a divisorial contraction. In particular, it a step of the MMP of $\overline{M}_g$.

If the F-conjecture is true, this is the only possible first step of the MMP of $\overline{M}_g$.

For every $i \in \{0, 2, 3, \ldots, \lfloor \frac{g}{2} \rfloor\}$, we are going to define the notions of i-stability and i^+-stability for a curve C of genus g.

(In our work, we carry out a more general analysis considering the moduli space of pointed curves $\overline{M}_{g,n}$, and more general stability conditions. For the sake of the exposition, we do not cover these topics in this talk.)

First, some preliminary notions:

We say that an irreducible component E of C is an elliptic bridge if E has genus one, and it is attached to $C \setminus E$ with two nodes. The type of E is "0" if $C \setminus E$ is connected. The type is "i" if $C \setminus E$ is not connected and one of the two components has genus i.

A tacnode p on a curve C is of type "0" if $C \setminus p$ is connected. It is of type "i" if $C \setminus p$ is not connected and one of the two components has genus i.

Now, we introduce the actual stability conditions:

A curve C is i-stable if the canonical bundle is ample, it has at most nodes, cusps and tacnodes as singularities, all tacnodes are of type i, and it has not attached elliptic tails.

A curve C is i^+-stable if it is i-stable and moreover it does not contain elliptic bridge of type i.

Using these stability notions, we can prove the following result.

Theorem (C., Tasin , Viviani)**.** *For every i in $\{0, 2, \ldots, \lfloor \frac{g}{2} \rfloor\}$, the moduli spaces M_g^i and $M_g^{i^+}$ of i-stable and i^+-stable curves are projective varieties.*

The natural morphisms

$$\begin{array}{ccc} M_g^{ps} & \dashrightarrow & M_g^{i^+} \\ & \searrow \quad \swarrow & \\ & M_g^i & \end{array}$$

form flips, so they are possible steps of the MMP for $\overline{M}_g$.

These flips replace the locus parametrizing curves containing elliptic bridge of type i with the locus parametrizing curves containing a tacnodes of type i.

If the F-conjecture is true, these are the only possible second steps for the MMP of $\overline{M}_g$.

Two consequences of our result are:

- the above defined stability conditions are not random definitions, but they are dictated by the MMP of $\overline{M}_g$,
- we can write down a partial explicit Shokurov decomposition for $\overline{M}_g$. By definition, this is the decomposition of the polytope of adjoint divisors according to their ample model.

Theorem (C., Tasin, Viviani)**.** *Let L be an adjoint $\mathbb{Q}$-divisor on $\overline{M}_g$ written as*

$$L = K_{\overline{M}_g} + a\lambda + \alpha_0\delta_0 + \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \alpha_i \delta_i$$

with $a \geq 0$, $0 \leq \alpha_i \leq 1$ for all i, $3|\alpha_i - \alpha_j| < 1$ for all i and j different from 0, and such that if $\alpha_0 = 1$ then $\alpha_i > 0$ for all $i \neq 0$.

Furthermore, assume that

$$7 - a \leq 10\alpha_0, \tag{1}$$

$$7 - a + \alpha_i + \alpha_{i+1,} \leq 12\alpha_0 \textit{ for all } i \neq 0, 1 \tag{2}$$

Then the ample model of L is

- *$\overline{M}_g$ itself, i.e. L is ample, if $9 - a + \alpha_1 < 12\alpha_0$;*
- *$\overline{M}_g \to M_g^{ps}$ if $9 - a < 12\alpha_0 \leq 9 - a + \alpha_1$, or $12\alpha_0 \leq 9 - a$, and both Equations (1) and (2) are strict;*
- *$\overline{M}_g \to M_g^0$ if $12\alpha_0 \leq 9 - a$, equality holds in Equation (1) but does not hold in Equations (2) for all i;*
- *$\overline{M}_g \to M_g^i$ with $i \neq 0$ if $12\alpha_0 \leq 9 - a$, equality does not hold in Equation (1), and it holds in Equations (2) only for that value of i.*

Using more general stability notions we are able to cover more cases, e.g. when equality in Equation (2) holds for multiple values of i, but I hope these results are enough to encourage you to

Keep on running MMP and crossing walls

THANK YOU FOR THE ATTENTION